Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00031-9

J. Appl. Maths Mechs, Vol. 64, No. 1, pp. 111–115, 2000
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0021-8928/00/$—see front matter

SIMPLE EXPRESSION FOR GREEN'S FUNCTION OF THE PROBLEM OF SHIP WAVES IN A DEEP HOMOGENEOUS LIQUID†

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(*Received* 22 *October* 1999)

The classical problem of steady ship waves which are induced in a deep liquid by a domain of surface pressures that is moving uniformly and rectilinearly is investigated. Exact expressions are known (in the linear formulation) for the characteristics of the induced waves in the form of double integrals and single integrals with a special function, the integral exponent. Using the analytic properties of the solution, a simpler expression is obtained for the elevation of the surface of the liquid in the form of single integrals of standard functions, which enables one to simplify considerably the numerical analysis of the neighbouring wave field domain. © 2000 Elsevier Science Ltd. All rights reserved.

The well-known solution of the linear problem of surface ship waves [1, 2] represents the elevation of the surface of the liquid as a sum of two terms. One of these (in the form of a double integral) describes the local perturbations in the neighbourhood of the generator while the other (in the form of a single integral) describes the system of ship waves behind the generator. A double integral which has an unbounded domain of integration and an oscillating integrand is the most difficult for calculations, and, on account of this, it is of interest in to transform this term into a form which is more convenient for calculations. A significant advance in this direction has been its transformation into a single integral with a special function, the integral exponent [3]. Such a technique has turned out to be extremely constructive: its use has enabled an analogous solution to be obtained in problems on surface waves in a liquid of finite depth [4] and internal waves in a stratified liquid [5]. However, calculation of the local term takes up about 90% of the time required for the calculations [3], which is due to the accompanying calculations of the integral exponent with a complex argument. Subsequent investigations of the problem of internal ship waves [6] have shown that the use of the analytic properties of the solution enables one to transform the double integral into a single integral of standard functions and thereby substantially reduce the time required to calculate the local term of the perturbation field. This result, obtained for internal ship waves, cannot be immediately transferred to the surface wave model since there are differences in the dispersion relations of the internal and surface waves associated with the different boundary conditions on the surface of the liquid used in these models. However, in the case of surface waves, local perturbations as well as wave perturbations can be represented in the form of single integrals with a simple integrand, and a form of representation of the solution is thereby obtained which is economic for calculations.

1. Suppose that pressures of the form move over the surface $z = 0$ of a deep liquid, occupying a domain $-\infty < x_1, y < +\infty, -\infty < z < 0$, at a constant velocity c in the negative direction of the x axis.

$$P_a = p_0 f(x_1 + ct,\ y) \tag{1.1}$$

In the linear formulation, the perturbations of the liquid, which are created by pressures (1.1), are described by a Laplace equation in the potential of the perturbed velocities

$$\Delta\phi = 0 \ (-\infty < x,\ \ y < +\infty,\ -\infty < z < 0) \tag{1.2}$$

with the boundary conditions

$$\begin{aligned} &\phi_{xx} + \varepsilon\phi_x + \beta\phi_z = -(\rho c)^{-1} P_{ax} \ (z = 0) \\ &\phi \to 0 \ (|x|,\ |y| \to \infty,\ \ z \to -\infty) \end{aligned} \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 1, pp. 115–120, 2000.

Here, $x = x_1 + ct$, $\beta = gc^{-2}$, g is the acceleration due to gravity and ρ is the liquid density. The equation of the surface of the liquid is specified by the formula

$$\zeta = -(\rho g)^{-1} P_a - cg^{-1}(\phi_x + \varepsilon\phi) \quad (z = 0) \tag{1.4}$$

where ε is the Rayleigh parameter ($\varepsilon \to +0$).

Equations (1.2) – (1.4) are written in the system of coordinates associated with pressures (1.1).

2. By carrying out a Fourier integral transformation with respect to the horizontal coordinates x and y on (1.2) – (1.4), we obtain an expression for the elevation of the surface of the liquid

$$\zeta(x, y) = \frac{p_0\beta}{4\pi^2\rho g}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\exp[i(mx+ny)]}{m^2 - im\varepsilon - \beta k}F(m, n)\,dm\,dn \tag{2.1}$$

Here, $k = \sqrt{m^2 + n^2}$; $F(m, n)$ is the Fourier transform of the function $f(x, y)$.

We now consider a model distribution of the pressures of the form

$$f(x, y) = (R^2/a^2 + 1)^{-3/2}, \quad F(m, n) = 2\pi a^2 e^{-ak}, \quad R^2 = x^2 + y^2 \tag{2.2}$$

When the parameter a tends to zero and the product $p_0 a^2$ tends to $(2\pi)^{-1}$, this distribution degenerates into a δ-function which enables one to obtain an expression for Green's function of the problem in question. From this point of view, the introduction of function (2.2) can be considered as a regularization of (2.1). A pressure which is constant and equal to p_0 within a circle of radius a and equal to zero outside this circle has previously been used [1, 2] with the same purpose. The Fourier transform of this distribution is expressed in terms of a Bessel function which is less convenient than (2.2).

We now substitute expression (2.2) into (2.1) and, changing to polar coordinates

$$(m, n) = k(\cos\theta, \sin\theta), \quad (x, y) = R(\cos\gamma, \sin\gamma)$$

we thus obtain

$$\zeta(x, y) = \frac{P}{2\pi}\int_0^{2\pi}\frac{1}{\cos^2\theta}\int_0^{\infty}\frac{k\exp[k(-a+i\mu)]}{k - (\beta + i\varepsilon\cos\theta)\cos^{-2}\theta}\,dk\,d\theta =$$
$$= -\frac{P}{\pi}\frac{\partial}{\partial a}\mathrm{Re}\int_{-\pi/2}^{\pi/2}\frac{J_\varepsilon(\theta, R, \gamma)}{\cos^2\theta}\,d\theta, \quad P = p_0\frac{a^2\beta}{\rho g}, \quad \mu = R\cos(\theta - \gamma) \tag{2.3}$$

$$J_\varepsilon(\theta, R, \gamma) = \int_0^{\infty}\frac{\exp[k(-a+i\mu)]}{k - (\beta + i\varepsilon\cos\theta)\cos^{-2}\theta}\,dk \tag{2.4}$$

3. When $\varepsilon > 0$, the inner integral (2.4) has a pole in the upper complex half-plane of the variable k. As was done previously [3], using the methods of contour integration and the change of variable

$$k[a + i\mu] = \tau, \quad \mathrm{Im}\,\tau = 0, \quad \mathrm{Re}\,\tau \quad 0$$

we transform integral (2.4) to the form

$$J_\varepsilon(\theta, R, \gamma) = 2\pi i H(\cos(\theta - \gamma))\exp(\psi_\varepsilon) + \int_0^{\infty}\frac{\exp(-\tau)}{\tau + \psi_\varepsilon}\,d\tau$$
$$\psi_\varepsilon = (-a + i\mu)(\beta + i\varepsilon\cos\theta)\cos^{-2}\theta, \quad H(\eta) = \begin{cases}1, & \eta > 0\\ 0, & \eta < 0\end{cases} \tag{3.1}$$

In the expression on the right-hand side of (3.1), it is possible to take the limit as $\varepsilon \to +0$, and, on doing this and using the definition in [7] of the integral exponent $E_1(\eta)$, we obtain

$$\zeta(x, y) = -\frac{P}{\pi}\mathrm{Re}\frac{\partial}{\partial a}\int_{-\pi/2}^{\pi/2}\Psi_2[2\pi i H(\mu) + E_1(\psi)]\,d\theta \tag{3.2}$$

$$\Psi_n = \exp(\psi)\cos^{-n}\theta, \quad \psi = \psi_\varepsilon|_{\varepsilon=0} = (-a + i\mu)\beta\cos^2\theta$$

Formula (3.2) gives a representation for $\zeta(x,y)$ in the form of a single integral with a special function in the integrand which, in calculations of the wave field using formula (3.2), has to be computed for complex values of its argument and involves a substantial volume of calculations [3].

4. In the case of the internal wave field, an expression similar to (3.2) has been transformed [6] into a formula containing single integrals of standard functions. This method cannot be directly used to transform relation (3.2) into a simpler formula due to differences in the dispersion relations of the surface and internal waves. However, integral (3.2) can be simplified considerably by making use of the analytic properties of the integral exponent $E_1(\eta)$.

We will use the fact that $E_1(\eta)$ has a logarithmic singularity at zero and at infinity. The difference in the values of $E_1(\eta)$ on the sides of a cut along the real axis of the variable η from $-\infty$ to 0 is calculated using the formula

$$E_1(-\eta+i0)-E_1(-\eta-i0) = -2\pi i$$

The asymptotic born of $E_1(\eta)$ when $|\eta| \to \infty$ has the form [7]

$$E_1(\eta)=\eta^{-1}\exp(-\eta)(1+O(\eta^{-1}))\left(|\arg\eta|<\frac{3}{2}\pi\right) \tag{4.1}$$

The substitution $x = -x'$ and the replacement $\theta = -\theta'$ in (3.2) shows that

$$\zeta(-x,\ y)=\zeta(x,\ y)-2P\operatorname{Im}\frac{\partial}{\partial a}\int_{-\pi/2}^{\pi/2}\Psi_2 d\theta \tag{4.2}$$

It is therefore possible in the subsequent transformations to confine the treatment to the case when $x \geqslant 0$.

We separate from (3.2) the term containing the special function E_1

$$J_1=\operatorname{Re}\int_{-\pi/2}^{\pi/2}\Psi_2 E_1(\psi)d\theta \tag{4.3}$$

Consider the strip $|\operatorname{Re}\theta| \leqslant \pi/2$ in the complex plane of the integration parameter θ. The poles of the function ψ, $E(\theta) = E_1(\psi)$ and $\theta = \mp\pi/2$ are the singular points of the function $|\operatorname{Im}\theta| = \infty$, and $\theta = \theta_0$ are the zeros of ψ, where θ_0 is the solution of the equation

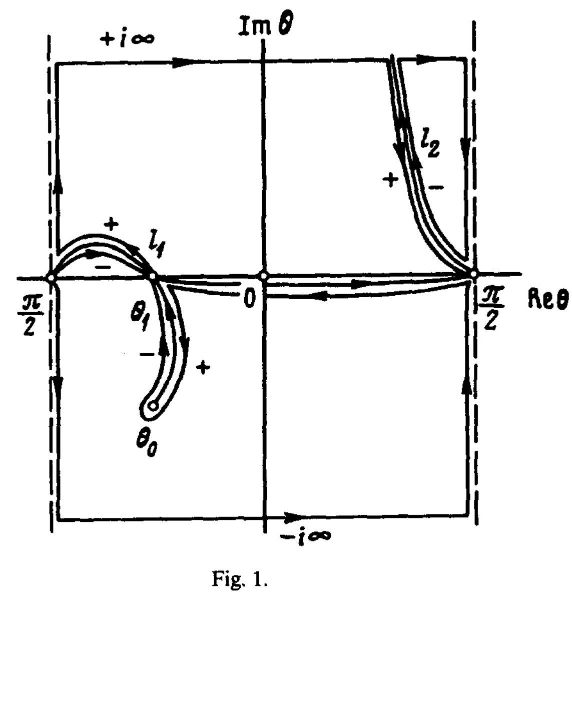

Fig. 1.

$$a - iR\cos(\theta-\gamma)=0: \ \ \mathrm{Re}\,\theta_0 = -\pi/2+\gamma, \ \ \mathrm{sh}(\mathrm{Im}\,\theta_0) = -a/R \tag{4.4}$$

In order to separate out the single-valued branch of the integrand of (4.3), we construct cuts from θ_0 to $-\pi/2$ as long the curve l_1 and from $\pi/2$ to Im $\theta = +i\infty$ along the curve l_2 such that Re $\psi < 0$ and Im $\psi = 0$ on these curves (Fig. 1).

We note that the curve l_1 intersects the real axis θ at the point $\theta_1 = -\pi/2 + \gamma$. We denote the part of l_1 which goes from θ_0 to θ_1 by l_{11} and the part which goes from θ_1 to $-\pi/2$ by l_{12}. In addition, we denote the sides of the cuts on which $l_{11}^{\pm}$, $l_{12}^{\pm}$ has the corresponding sign (Fig. 1) by $l_2^{\pm}$. Analysis shows that, on moving along Im ψ from l_1 to θ_0, the imaginary part of ψ is positive to the right of the curve and, on moving along $-\pi/2$ from l_2 to $\pi/2$, Im $\theta = +i\infty$ it is positive to the left of the curve. The asymptotics (4.1) enable one to take the limit with the closure of the contours of integration in the neighbourhoods of the singular points.

By transforming the paths of integration in (4.3) first into the upper and then into the lower complex half-planes of the variable of integration θ, we obtain

$$J_1 = \mathrm{Re}\left(\int_{-\pi/2}^{\theta_1} + \int_{\theta_1}^{\pi/2}\right) = \mathrm{Re}\left(-\int_{l_{12}^-} + \int_{l_{12}^+} + \int_{-\pi/2}^{-\pi/2+i\infty} - \int_{l_2^+}\right) \tag{4.5}$$

$$J_1 = \mathrm{Re}\left(\int_{-\pi/2}^{\theta_1} + \int_{\theta_1}^{\pi/2}\right) = \mathrm{Re}\left(\int_{-\pi/2}^{-\pi/2-i\infty} + \int_{\pi/2-i\infty}^{\pi/2} - \int_{l_{11}^+} + \int_{l_{11}^-}\right) \tag{4.6}$$

It is easily verified that

$$\mathrm{Re}\left(\int_{-\pi/2}^{-\pi/2+i\infty} + \int_{\pi/2-i\infty}^{\pi/2}\right) = 0 \text{ and } \mathrm{Re}\int_{-\pi/2}^{-\pi/2-i\infty} = \mathrm{Re}\int_{\pi/2}^{\pi/2+i\infty} = \mathrm{Re}\int_{l_2^-}$$

Adding (4.5) and (4.6), we find that

$$2J_1 = \mathrm{Re}\left[\left(\int_{l_{12}^+} - \int_{l_{12}^-}\right) - \left(\int_{l_{11}^+} - \int_{l_{11}^-}\right) - \left(\int_{l_2^+} - \int_{l_2^-}\right)\right] \tag{4.7}$$

Next, taking account of the difference in the values of the integral exponent on the sides of the cuts, we obtain from (4.7) that

$$J_1 = \pi\,\mathrm{Im}\left(\int_{l_{12}} - \int_{l_{11}} - \int_{l_2}\right)\Psi_2 d\theta \tag{4.8}$$

Substituting expression (4.8) into (3.2), we drive an expression for $\zeta(x, y)$ in the form of a sum of single integrals

$$\zeta(x,y) = P\,\mathrm{Im}\frac{\partial}{\partial a}\left(2\int_{\theta_1}^{\pi/2} + \int_{l_{11}} - \int_{l_{12}} + \int_{l_2}\right)\Psi_2 d\theta \tag{4.9}$$

We differentiate formula (4.9) is differentiated with respect to a taking account of the fact that, in the second integral in (4.9), the lower limit of integration depends on a and, therefore,

$$I_2 = \frac{\partial}{\partial a}\int_{l_{11}} \Psi_2 d\theta = \frac{\partial}{\partial a}\int_{\theta_0,\,\theta\in l_{11}}^{\theta_1} \Psi_2 d\theta = -\frac{\partial\theta_0/\partial a}{\cos^2\theta_0} - \int_{l_{11}} \psi_a \Psi_2 d\theta$$

We find from (4.4) that $\partial\theta_0/\partial a = i[R\sin(\theta_0-\gamma)]^{-1}$. Next, on carrying out the necessary transformations and transforming the path of integration in the domain Re $\psi \leqslant 0$, we obtain the final expression for the elevation of the surface of the liquid in the form of a sum of single integrals of standard functions

$$\zeta(x,\ y) = -P\beta[\operatorname{sgn}(x)\zeta_1(|x|,\ y) + \zeta_2(|x|,\ y)] \tag{4.10}$$

$$\zeta_1(x, y) = \operatorname{Im} \int_{-\pi/2}^{\pi/2} \Psi_4 d\theta \tag{4.11}$$

$$\zeta_2(x,\ y) = \frac{a^2x^2 - y^2(R^2 + a^2)}{\beta\sqrt{R^2 + a^2}(y^2 + a^2)^2} + \operatorname{Im} \int_{\theta_0}^{+i\infty} \Psi_4 d\theta \tag{4.12}$$

The integration path in (4.11) assess along the real axis while, in (4.12), the path must not pass through the points $\theta = \pm\pi/2$.

Expression (4.12) can be modified, if partial integration is carried out. The contribution of the boundary point θ_0 is opposite in sign to the first term of this expression and cancels it. However, the integrand is complicated in this case and it is required that the integration path should not pass through the stationary points.

The integral from (4.12) can be transformed to a form which is more convenient for calculations. We choose the path of integration in (4.12) to be parallel to the imaginary axis and, putting $\theta = \theta_r + i\theta_i$, we make the substitution $\cos\omega = 1/\operatorname{ch}\theta_i$. As a result we obtain the formula

$$\operatorname{Im} \int_{\theta_0}^{+i\infty} \frac{\exp(-\psi)}{\cos^4\theta} d\theta = \operatorname{Re} \int_{\omega_0}^{\pi/2} \frac{\cos^3\omega}{d^4(\omega)} \exp\left[-\beta \frac{\cos\omega(a\cos\omega + R\sin\omega)}{d^2(\omega)}\right] d\omega \tag{4.13}$$

$$d(\omega) = \sin\gamma + i\sin\omega\cos\gamma,\quad \omega_0 = -0\arcsin(a/\sqrt{a^2 + R^2})$$

This transformation is possible when $\gamma \leqslant \gamma_0 > -\pi/2$, when the rectilinear integration path does not pass through the singular point $\theta = -\pi/2$. When $-\pi/2 \leqslant \gamma < \gamma_0$, the path of integration in (4.12) is constructed from a segment parallel to the real axis θ from θ_0 to the imaginary axis and the interval of the imaginary axis from $i \operatorname{Im} \theta_0$ to $i\infty$. The integral along the imaginary axis is transformed to a form, analogous to (4.13).

If one puts $a = 0$ in formulae (4.11) and (4.12), and the product p_0a^2 in (4.10) is replaced by $(2\pi)^{-1}$, an expression is obtained for the waves formed by pressures which are concentrated at a single point of the surface of the liquid.

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Translated by E.L.S.